How Small is Small?

Asymptotic Series in Physics

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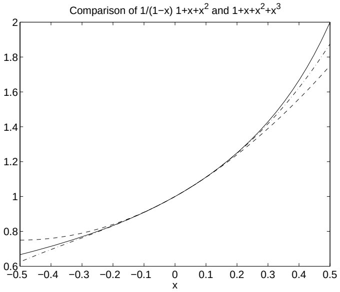

Abstract

How often have you used the small angle approximation $\sin\theta \approx \theta$ or the binomial expansion $(1+x)^p \approx 1 + px$? How accurate is the formula $T = 2\pi\sqrt{\frac{L}{g}}$ giving the period of a pendulum for angles that are not small? How fast must you have to move for relativistic effects to be important? Do series expansions have to converge to be useful? In this lecture we will explore some of these questions as we investigate the role of series expansions in our approximations. In particular, we describe the typical use of series expansions in undergraduate physics, provide some examples and if there is time we may see how divergent series are often useful.

Outline of Talk

Geometric Series

Definition

A geometric series is of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^n + \dots. \tag{1}$$

Here a is the first term and r is called the ratio.

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Examples

$$\sum_{n=0}^{\infty} \frac{2^n}{3^n} = 1 + \frac{2}{3} + \frac{2^2}{3^2} + \frac{2^3}{3^3} + \cdots$$

$$\sum_{n=2}^{\infty} 3(2^n) = 3(2^2) + 3(2^3) + 3(2^4) + \cdots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$$

nth Partial Sum

Sum of Geometric Progression

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- Consider the nth partial sum:

$$s_n = a + ar + \cdots + ar^{n-2} + ar^{n-1}. \tag{2}$$

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- Subtracting

$$(1-r)s_n = a - ar^n. \tag{4}$$

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$$(1 - r)s_n = a - ar^n. \tag{4}$$

- Thus, the nth partial sum is

$$s_n = \frac{a(1 - r^n)}{1 - r}. \tag{5}$$

Limit of Partial Sums

Recall that the sum, if it exists, is given by

$$S = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{a(1 - r^n)}{1 - r}.$$

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Geometric Series Result

The sum of the geometric series is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \qquad |r| < 1. \tag{6}$$

Examples

Example 1. $\sum_{n=0}^{\infty} \frac{1}{2^n}$

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Example 2. $\sum_{k=2}^{\infty} \frac{4}{3^k}$

In this example we note that the first term occurs for $k = 2$. So, $a = \frac{4}{9}$. Also, $r = \frac{1}{3}$. So,

$$S = \frac{\frac{4}{9}}{1 - \frac{1}{3}} = \frac{2}{3}.$$

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More General Expressions

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$$\frac{1}{\sqrt{r^2+R^2}} = (r^2+R^2)^{-1/2} = \frac{1}{R}\left(1+\left(\frac{r}{R}\right)^2\right)^{-1/2} \approx \frac{1}{R}\left(1 - \frac{r^2}{2R^2} + \frac{3}{8}\frac{r^4}{R^4}\right)$$

Looking for Patterns ...

Consider $(a+b)^p$ for nonnegative integer p's:

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Do you see any patterns?

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So, we can write the $k+1$st term in the expansion as

$$C^r_{k-1} a^{n-k} b^k.$$

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Example - 6th term in $(a+b)^{51}$

$a^{51-5}b^5 = a^{46}b^5$. What is the numerical coefficient?

Patterns in the Coefficients - Pascal's Triangle

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array} \tag{8}$$

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- Each row begins and ends with a one.

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 & & & \searrow\ \swarrow & & \searrow\ \swarrow & & \\
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\end{array}
\tag{10}
$$

We can generate the next several rows of our triangle.

$$
\begin{array}{lccccccccccccc}
n=3: & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
\tag{11}
$$

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The kth term in the expansion of $(a+b)^n$.

Let $r = k-1$. Then this term is of the form $C_r^n a^{n-r} b^r$, where $C_r^n = C_r^{n-1} + C_{r-1}^{n-1}$.

Combinatorial Coefficients

Actually, the coefficients have been found to take a simple form.

$$C_r^n = \frac{n!}{(n-r)!r!} = \left(\begin{array}{c} n \\ r \end{array} \right).$$

For example, the $r = 2$ case for $n = 4$ involves the six products: *aabb*, *abab*, *abba*, *baab*, *baba*, and *bbaa*.

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The Binomial Series for Nonnegative Integer Powers

So, we have found that

$$(a+b)^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right) a^{n-r} b^r. \tag{12}$$

Note: Sums may be infinite.

$$(1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + \ldots .$$

Binomial Expansion - Other Powers

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$$(1+x)^p = \sum_{r=0}^{\infty} \begin{pmatrix} p \\ r \end{pmatrix} x^r \qquad \text{for } p \text{ real.} \tag{13}$$

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$$(1+x)^p = \sum_{r=0}^{\infty} \binom{p}{r} x^r \qquad \text{for } p \text{ real.} \tag{13}$$

Is There a Problem?

Consider the coefficient for $r = 1$ in an expansion of $(1+x)^{-1}$.

$$\binom{-1}{1} = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

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$$\left(\begin{array}{c} -1 \\ 1 \end{array} \right) = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

where $(-1)! = (-1)(-2)(-3)\cdots =????$

Eliminate the factorials

Exercising a little care:

$$\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.$$

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In General

$$\begin{aligned} \binom{p}{r} &= \frac{p!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)}{r!}. \end{aligned} \tag{14}$$

Theorem

The general binomial expansion of $(1+x)^p$ for p real is

$$(1+x)^p = \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!}x^r. \tag{15}$$

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Approximations

Often we need the first few terms for the case that $x \ll 1$:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2} x^2 + O(x^3). \tag{16}$$

Example

Relativistic Kinetic Energy

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&= \left(\left(1+\left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right)+\cdots\right)-1\right)mc^2 \\
&= \frac{1}{2}\left(\frac{v^2}{c^2}\right)mc^2 \\
&= \frac{1}{2}mv^2.
\end{aligned}
$$

Power Series

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Taylor Series

A *Taylor series* expansion of $f(x)$ about $x = a$ is the series

$$f(x) \sim \sum_{n=0}^{\infty} c_n (x-a)^n, \tag{17}$$

where

$$c_n = \frac{f^{(n)}(a)}{n!}. \tag{18}$$

Taylor and Maclaurin Series

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Maclaurin Series - $a = 0$

Sample Coefficient Computation

Expand $f(x) = e^x$ about $x = 0$.

n	$f^{(n)}(x)$	$c_n = \frac{f^{(n)}(0)}{n!}$
0	e^x	$\frac{e^0}{0!} = 1$
1	e^x	$\frac{e^0}{1!} = 1$
2	e^x	$\frac{e^0}{2!} = \frac{1}{2!}$
3	e^x	$\frac{e^0}{3!} = \frac{1}{3!}$

In this case, we have that the pattern is obvious:

$$c_n = \frac{1}{n!}.$$

So,

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

More Expansions

Common Series Expansions

$$
\begin{aligned}
\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \\
\tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\
\ln(1+x) &= -x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}
\end{aligned}
$$

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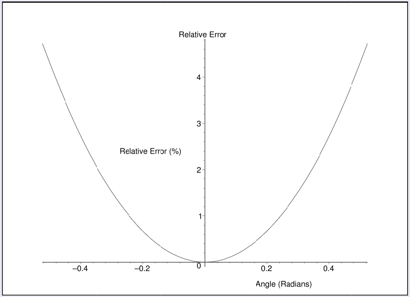
$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots \qquad \Rightarrow \qquad \text{Relative Error } = \left\| \frac{\sin\theta - \theta}{\sin\theta} \right\|.$$

Small Angle Approximation

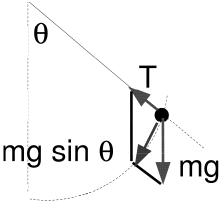
$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots \qquad \Rightarrow \qquad \text{Relative Error } = \left\| \frac{\sin\theta - \theta}{\sin\theta} \right\|.$$

The relative error in percent when approximating $\sin\theta$ by θ.



A one percent relative error occurs for $\theta \approx 0.24$ radians $= 0.24\text{rad}\ \frac{180^o}{\pi\text{rad}} < 14^o$.

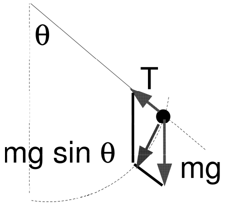
The Simple Pendulum



The Simple Pendulum

Consider Newton's Second Law for Rotational Motion:

$$\tau = I\alpha$$



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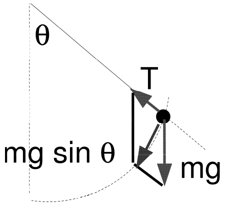

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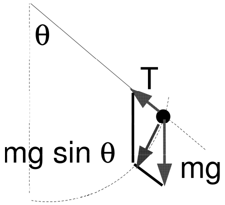

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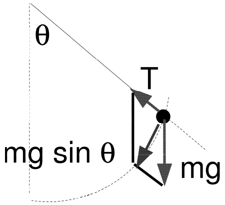

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- Linear Pendulum: $L\ddot{\theta} + g\theta = 0.$

Linear Pendulum, $L\ddot{\theta} + g\theta = 0$

The general solution

$$\theta(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) \tag{19}$$

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The period is found to be

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}. \tag{20}$$

The Nonlinear Pendulum, $\ddot{\theta} + \omega^2 \sin\theta = 0$

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Multiply Equation by $\dot{\theta}$:

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Rearrange and integrate:

$$t = \int dt = \int \frac{d\theta}{\sqrt{2(c + \omega^2 \cos\theta)}}.$$

Energy Analysis

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$$T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2.$$

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and for $\theta_{\text{max}} = \theta_0$,

$$E = mgL(1 - \cos\theta_0) = mL^2\omega^2(1 - \cos\theta_0).$$

Returning to Nonlinear Pendulum Computation

Therefore, we have found that

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$$\frac{1}{2}\dot{\theta}^2 = 2\omega^2 \left[\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}\right]. \tag{24}$$

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$$\frac{d\theta}{dt} = 2\omega\left[\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right]^{1/2}. \tag{25}$$

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And separating:

$$2\omega dt = \frac{d\theta}{\left[\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right]^{1/2}}$$

The Period of Oscillation

Consider a quarter of a cycle ($\theta = 0$ to $\theta = \theta_0$):

$$T = \frac{2}{\omega} \int_0^{\theta_0} \frac{d\phi}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}}. \tag{26}$$

Defining $z = \frac{\sin \frac{\theta}{2}}{\sin \frac{\theta_0}{2}}$ and $k = \sin \frac{\theta_0}{2}$, we obtain

$$\boxed{T = \frac{4}{\omega} \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}}}$$

This is done using

$dz = \frac{1}{2k} \cos \frac{\theta}{2}\, d\theta = \frac{1}{2k}(1 - k^2 z^2)^{1/2}\, d\theta$ and $\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} = k^2(1 - z^2)$.

Period - Small Angle Approximation

For small angles, $k = \sin\frac{\theta_0}{2}$ is small.

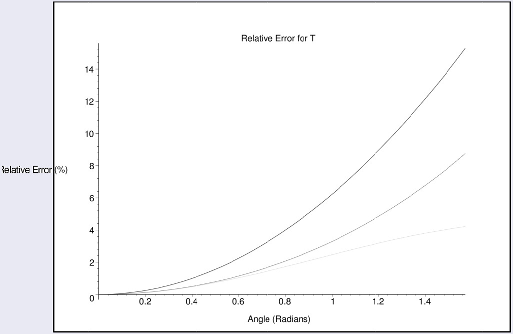
$$(1-k^2z^2)^{-1/2} = 1 + \frac{1}{2}k^2z^2 + \frac{3}{8}k^2z^4 + O((kz)^6)$$

$$\begin{aligned} T &= \frac{4}{\omega}\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \\ &= 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1}{4}k^2+\frac{9}{64}k^4+\dots\right]. \end{aligned} \tag{27}$$

… and finally!

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \dots\right]$$

The Relative Error - Using 1,2,3 Terms



Usefulness of Divergent Series

Convergent Power Series

To date you have learned that convergent power series are good and divergent series are bad. Recall the ratio test: For $\sum c_n(x-a)^n$, the ratio test

$$\rho = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| |x-a| < 1$$

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Simple Example

Recall

$$\ln(1+x) = -x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$$

This converges absolutely when

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} n}{(-1)^n (n+1)} \right| |x| < 1,$$

or $|x| < 1$.

Asymptotic Series

Definition

$f(x) \simeq \phi(x) \sum_{n=0}^{\infty} \frac{a_n}{x^n}$ provided

$$\lim_{|x|\to\infty} x^n \left[\frac{f(x)}{\phi(x)} - \sum_{n=0}^{N} \frac{a_n}{x^n} \right] \to 0$$

Thus,

- for a given N, the sum of $N+1$ terms of the series can be as close to $\frac{f(x)}{\phi(x)}$ as one desires for sufficiently large x.
- For each x and N the error is of the order $1/x^{N+1}$
- However, the series is divergent and thus there are an optimal number of terms needed.

A Simple Asymptotic Series

Example

$$\int_0^\infty \frac{e^{-zt}}{1+t^2}dt = \frac{1}{z} - \frac{2!}{z^3} + \frac{4!}{z^5} - \cdots + \frac{(-1)^{n-1}(2n-2)!}{z^{2n-1}} + R_n(z)$$

where $|R_n(z)| \leq \frac{(2n)!}{z^{2n+1}}$ and for fixed n, $\lim_{|z|\to\infty} R_n(z) = 0$.

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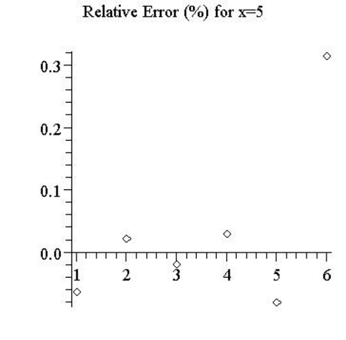

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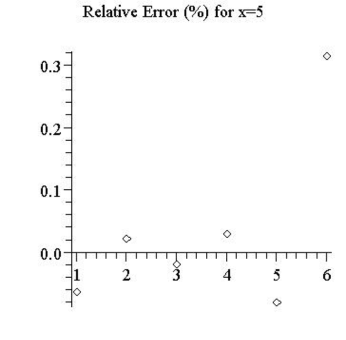

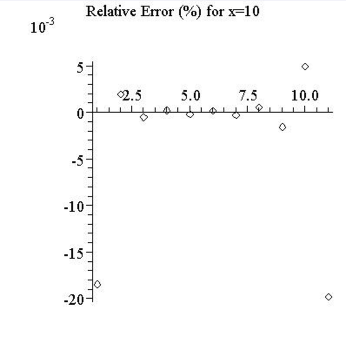

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In general,

$$Ei(x) = \frac{e^{-x}}{x} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^n}$$

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Convergence of the Series $Ei(x) = \frac{e^{-x}}{x} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^n}$

$$\rho = \lim_{n \to \infty} \left| \frac{(n+1)!}{n!} \right| |x| = \infty,$$

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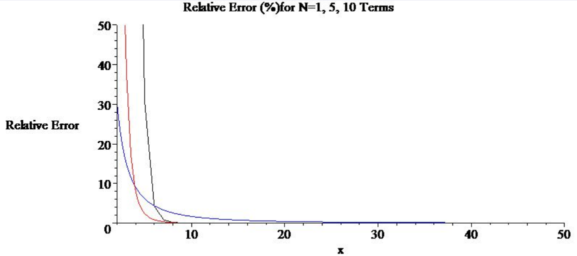
The Relative Error - Using 1,5,10 Terms



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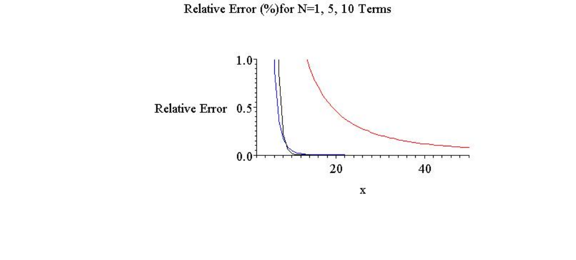
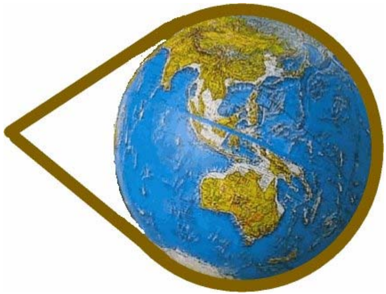


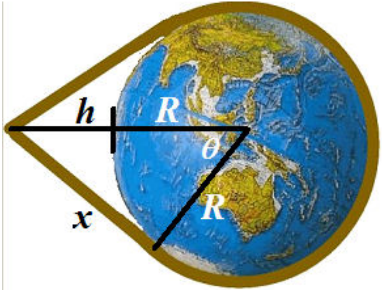
The World on a String Problem

The Problem

Tie a string around the Earths equator so that it is tight. Now, add ten feet to the string. Pull it at one point until it is tight but comes up to a point. How far from the Earths surface is this? (How long a pole would you need to support it?) To how many digits can you give your answer?



Key Equations



Hints

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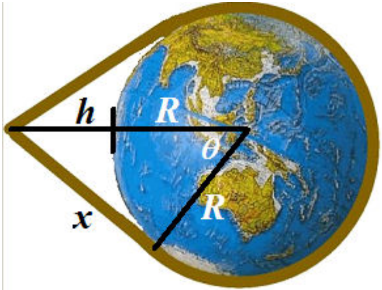
$$\tan\theta = \frac{x}{R}$$



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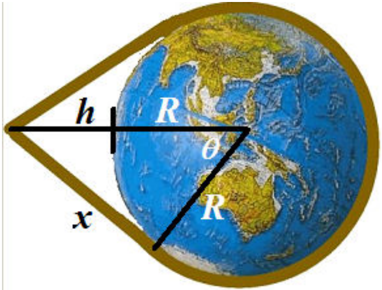

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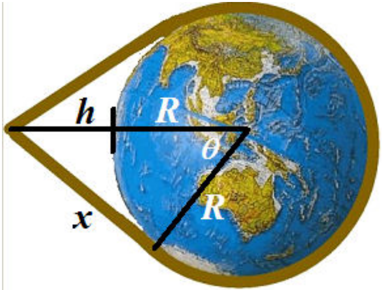

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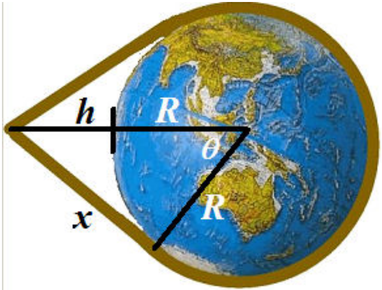

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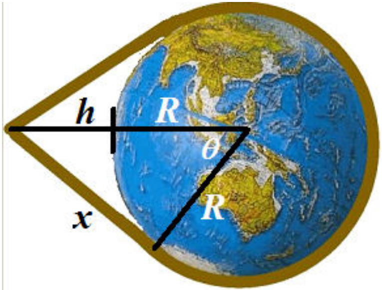

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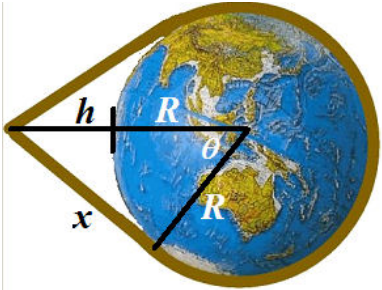

Summary of Talk

1. Geometric Series
2. Binomial Series
   - Special Relativity Example
3. Taylor Series
4. Applications
   - Small Angles
   - The Nonlinear Pendulum
5. Asymptotic Series
6. Homework - The World on a String